Supplementary Material of

"Automatic Design of Color Filter Arrays in The Frequency Domain"

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In this supplementary material, we prove the Theorem 1 which shows the solution to the following problem:

$$\begin{aligned}\mathbf{M}^{k+1} &= \underset{\mathbf{M}}{\operatorname{argmin}}\, \mathcal{L}(\mathbf{M}, \mathbf{N}_1^k, \mathbf{N}_2^k, \mathbf{S}^k, \mathbf{X}^k, \mathbf{x}^k, \mathbf{Y}^k, \mathbf{Z}^k) \\ &= \underset{\mathbf{M}}{\operatorname{argmin}}\, \|\mathbf{M}^{-1}\|_2 + \frac{\beta}{2}\|\mathbf{M} - (\mathbf{N}_1^k + i\mathbf{N}_2^k) + \mathbf{X}^k/\beta\|_F^2 \\ &= \underset{\mathbf{M}}{\operatorname{argmin}}\, \frac{1}{\beta}\|\mathbf{M}^{-1}\|_2 + \frac{1}{2}\|\mathbf{M} - \mathbf{W}^k\|_F^2.\end{aligned} \tag{1}$$

It is problem (17) in the main article.

Theorem 1. *The solution to problem* (1) *is:*

$$\mathbf{M}^{k+1} = \mathbf{U}^k \mathbf{\Sigma}^{k+1} (\mathbf{V}^k)^H, \tag{2}$$

where $\mathbf{U}^k \mathbf{\Lambda}^k (\mathbf{V}^k)^H$ is the SVD of $\mathbf{W}^k$, $\mathbf{U}^k$ and $\mathbf{V}^k$ are unitary matrices, $\mathbf{\Lambda}^k = diag(\boldsymbol{\lambda}^k)$, in which $diag(\mathbf{y})$ converts the vector $\mathbf{y}$ into a diagonal matrix whose j-th diagonal element is $\mathbf{y}_j$, $\boldsymbol{\lambda}^k = (\lambda_1^k, \lambda_2^k, \lambda_3^k)^T$ is the real vector of singular values of $\mathbf{W}^k$ and satisfies $\lambda_1^k \geq \lambda_2^k \geq \lambda_3^k > 0$, and $\mathbf{\Sigma}^{k+1} = diag(\boldsymbol{\sigma}^{k+1})$, in which $\boldsymbol{\sigma}^{k+1} = (\sigma_1^{k+1}, \sigma_2^{k+1}, \sigma_3^{k+1})^T$ is the solution to the following problem:

$$\min_{\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0} \; \frac{1}{\beta \sigma_3} + \frac{1}{2} \sum_{j=1}^3 (\sigma_j - \lambda_j^k)^2. \tag{3}$$

Before we prove it, we first quote the von Neumann's inequality [1]: Suppose $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices. Then $\langle \mathbf{A}, \mathbf{B} \rangle \leq \sum_j \delta_j(\mathbf{A}) \delta_j(\mathbf{B})$, where $\delta_j(\mathbf{B})$ is the j-th largest singular value of $\mathbf{B}$. The equality holds when the matrices of left and right singular vectors of $\mathbf{A}$ are the same as those of $\mathbf{B}$.

Proof.

$$\begin{aligned}&\frac{1}{\beta}\|\mathbf{M}^{-1}\|_2 + \frac{1}{2}\|\mathbf{M} - \mathbf{W}^k\|_F^2 \\ &= \frac{1}{\beta \delta_3(\mathbf{M})} + \frac{1}{2}(\|\mathbf{M}\|_F^2 - 2\langle \mathbf{M}, \mathbf{W}^k \rangle + \|\mathbf{W}^k\|_F^2) \\ &= \frac{1}{\beta \delta_3(\mathbf{M})} + \frac{1}{2}\left(\sum_{j=1}^3 \delta_j(\mathbf{M}) - 2\langle \mathbf{M}, \mathbf{W}^k \rangle + \sum_{j=1}^3 \delta_j(\mathbf{W}^k) \right) \\ &\geq \frac{1}{\beta \delta_3(\mathbf{M})} + \frac{1}{2} \sum_{j=1}^3 \left(\delta_j(\mathbf{M}) - 2\delta_j(\mathbf{M})\delta_j(\mathbf{W}^k) + \delta_j(\mathbf{W}^k) \right) \\ &= \frac{1}{\beta \delta_3(\mathbf{M})} + \frac{1}{2} \sum_{j=1}^3 \left(\delta_j(\mathbf{M}) - \delta_j(\mathbf{W}^k) \right)^2.\end{aligned}$$

According to the von Neumann's inequality, the equality can hold when the matrices of left and right singular vectors of $\mathbf{M}$ are the same as those of $\mathbf{W}^k$. Thus the theorem is proved. □

So by Theorem 1 the solving for $\mathbf{M}^{k+1}$ in problem (1) is converted into that for $\boldsymbol{\sigma}^{k+1}$ in (3), which is convex. In order to facilitate the presentation and calculation, we drop the superscript k of $\boldsymbol{\lambda}$ and reformulate (3) as:

$$\min_{\boldsymbol{\sigma}} \frac{1}{\beta \sigma_3} + \frac{1}{2}\|\boldsymbol{\sigma} - \boldsymbol{\lambda}\|_2^2, \text{s.t. } \mathbf{T}\boldsymbol{\sigma} \geq \mathbf{0}, \tag{4}$$

where $\mathbf{T} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

When applying ADM to (4), we first introduce auxiliary variables $\boldsymbol{\tau}$ and φ and rewrite it as:

$$\min_{\boldsymbol{\sigma},\boldsymbol{\tau},\varphi} \frac{1}{\beta\varphi}+\frac{1}{2}\|\boldsymbol{\sigma}-\boldsymbol{\lambda}\|_2^2+\mathcal{I}_{\mathbb{R}_+}(\boldsymbol{\tau}), \text{s.t. } \mathbf{T}\boldsymbol{\sigma}=\boldsymbol{\tau}, \varphi=\sigma_3. \tag{5}$$

The augmented Lagrangian function of (5) is:

$$\mathcal{L}_\sigma(\boldsymbol{\sigma},\boldsymbol{\tau},\varphi,\mathbf{u},v)=\frac{1}{\beta\varphi}+\frac{1}{2}\|\boldsymbol{\sigma}-\boldsymbol{\lambda}\|_2^2+\mathcal{I}_{\mathbb{R}_+}(\boldsymbol{\tau})+\langle\mathbf{u},\mathbf{T}\boldsymbol{\sigma}-\boldsymbol{\tau}\rangle+\langle v,\varphi-\sigma_3\rangle+\frac{\kappa}{2}\|\mathbf{T}\boldsymbol{\sigma}-\boldsymbol{\tau}\|_2^2+\frac{\kappa}{2}(\varphi-\sigma_3)^2, \tag{6}$$

where $\mathbf{u}$ and v are the Lagrange multipliers, and $\kappa>0$ is the penalty parameter which is fixed during the iterations.

Then by ADM problem (5) can be solved via the following iterations:

$$\begin{aligned}\boldsymbol{\sigma}^{t+1}&=\operatorname*{argmin}_{\boldsymbol{\sigma}}\mathcal{L}_\sigma(\boldsymbol{\sigma},\boldsymbol{\tau}^t,\varphi^t,\mathbf{u}^t,v^t)\\&=\operatorname*{argmin}_{\boldsymbol{\sigma}}\frac{1}{2}\|\boldsymbol{\sigma}-\boldsymbol{\lambda}\|_2^2+\frac{\kappa}{2}\|\mathbf{T}\boldsymbol{\sigma}-\boldsymbol{\tau}^t+\mathbf{u}^t/\kappa\|_2^2+\frac{\kappa}{2}(\varphi^t-\mathbf{d}^T\boldsymbol{\sigma}+v^t/\kappa)^2\\&=\mathbf{Q}^{-1}\left(\boldsymbol{\lambda}+\mathbf{T}^T(\kappa\boldsymbol{\tau}^t-\mathbf{u}^t)+\mathbf{d}(\kappa\varphi^t+v^t)\right),\end{aligned} \tag{7}$$

$$\begin{aligned}\boldsymbol{\tau}^{t+1}&=\operatorname*{argmin}_{\boldsymbol{\tau}}\mathcal{L}_\sigma(\boldsymbol{\sigma}^{t+1},\boldsymbol{\tau},\varphi^t,\mathbf{u}^t,v^t)\\&=\operatorname*{argmin}_{\boldsymbol{\tau}}\mathcal{I}_{\mathbb{R}_+}(\boldsymbol{\tau})+\frac{\kappa}{2}\|\mathbf{T}\boldsymbol{\sigma}^{t+1}-\boldsymbol{\tau}+\mathbf{u}^t/\kappa\|_2^2\\&=\max(\mathbf{0},\mathbf{T}\boldsymbol{\sigma}^{t+1}+\mathbf{u}^t/\kappa),\end{aligned} \tag{8}$$

$$\begin{aligned}\varphi^{t+1}&=\operatorname*{argmin}_{\varphi}\mathcal{L}_\sigma(\boldsymbol{\sigma}^{t+1},\boldsymbol{\tau}^{t+1},\varphi,\mathbf{u}^t,v^t)\\&=\operatorname*{argmin}_{\varphi}\frac{1}{\beta\varphi}+\frac{\kappa}{2}(\varphi-\sigma_3^{t+1}+v^t/\kappa)^2,\end{aligned} \tag{9}$$

$$\mathbf{u}^{t+1}=\mathbf{u}^t+\kappa(\mathbf{T}\boldsymbol{\sigma}^{t+1}-\boldsymbol{\tau}^{t+1}), \tag{10}$$

$$v^{t+1}=v^t+\kappa(\varphi^{t+1}-\sigma_3^{t+1}), \tag{11}$$

where $\mathbf{Q}=\mathbf{I}+\kappa\mathbf{T}^T\mathbf{T}+\kappa\mathbf{d}\mathbf{d}^T$, $\mathbf{d}=(0,0,1)^T$, and $\mathbf{I}\in\mathbb{R}^{3\times3}$ is the identity matrix.

Let $g^{t+1}=\sigma_3^{t+1}-v^t/\kappa$ in (9), then we have:

$$\varphi^{t+1}=\operatorname*{argmin}_{\varphi>0} q(\varphi)=\frac{1}{\beta\varphi}+\frac{\kappa}{2}(\varphi-g^{t+1})^2. \tag{12}$$

Since $q(\varphi)$ is differentiable w.r.t. φ on the set of positive real numbers, φ^{t+1} is to be among the positive real critical points of $q(\varphi)$, which are the positive real roots of the cubic equation $\varphi^3-g^{t+1}\varphi^2-1/(\beta\kappa)=0$. It has a closed-form solution and can be computed by the cubic formula.

The stopping criteria are:

$$\max\{\|\boldsymbol{\sigma}^{t+1}-\boldsymbol{\sigma}^t\|_\infty,\|\boldsymbol{\tau}^{t+1}-\boldsymbol{\tau}^t\|_\infty,\|\varphi^{t+1}-\varphi^t\|_\infty\}<\varepsilon_3 \tag{13}$$

$$\text{and}\quad \max\{\|\mathbf{T}\boldsymbol{\sigma}^{t+1}-\boldsymbol{\tau}^{t+1}\|_\infty,\|\varphi^{t+1}-\sigma_3^{t+1}\|_\infty\}<\varepsilon_4. \tag{14}$$

We summarize the whole solution process of problem (5) in Algorithm 1.

Algorithm 1 The ADM algorithm for problem (5)

Input: $\boldsymbol{\lambda}$, β, $\mathbf{T}$, $\kappa=1$, $\varepsilon_3=10^{-10}$, and $\varepsilon_4=10^{-10}$.
1: **Initialize:** $\boldsymbol{\tau}=\mathbf{0}$, $\varphi=0$, $\mathbf{u}=\mathbf{0}$, $v=0$, $t=0$.
2: **while** the stop conditions (13) and (14) are not met **do**
3: fix the others and update $\boldsymbol{\sigma}$ by (7).
4: fix the others and update $\boldsymbol{\tau}$ by (8).
5: fix the others and update φ by (9).
6: update the multipliers $\mathbf{u}$ and v by (10) and (11).
7: $t\leftarrow t+1$.
8: **end while**
Output: $\boldsymbol{\sigma}$.

References

[1] R. A. Horn and C. R. Johnson. *Matrix analysis.* Cambridge university press, 2012.