Supplementary Material of

"Automatic Design of High-Sensitivity Color Filter Arrays with Panchromatic Pixels"

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In the supplementary material, we prove Theorem 1 in Section 1. Then we present the solution process for problem (8) in Section 2.

1 Proof of Theorem 1

Theorem 1 shows the solution to the following problem:

$$\begin{aligned}\mathbf{M}^{k+1} &= \underset{\mathbf{M}}{\operatorname{argmin}}\, \mathcal{L}(\mathbf{M}, \mathbf{N}_1^k, \mathbf{N}_2^k, \mathbf{S}^k, \mathbf{X}^k, \mathbf{x}^k, \mathbf{Y}^k, \mathbf{Z}^k) \\ &= \underset{\mathbf{M}}{\operatorname{argmin}}\, \mathcal{I}_\Phi(\mathbf{M}) + \frac{\beta}{2}\|\mathbf{M} - (\mathbf{N}_1^k + i\mathbf{N}_2^k) + \mathbf{X}^k/\beta\|_F^2 \\ &= \underset{\mathbf{M}}{\operatorname{argmin}}\, \mathcal{I}_\Phi(\mathbf{M}) + \frac{1}{2}\|\mathbf{M} - \mathbf{W}^k\|_F^2,\end{aligned} \tag{29}$$

where $\Phi = \{\mathbf{M} | \|\mathbf{M}^{-1}\|_2 \le \nu\}$ and $\mathbf{W}^k = (\mathbf{N}_1^k + i\mathbf{N}_2^k) - \mathbf{X}^k/\beta$. The coefficient β is dropped in the last equality as this does not affect the solution. Problem (29) is problem (12) in the main body of the paper.

Theorem 1. *The solution to problem* (29) *is:*

$$\mathbf{M}^{k+1} = \mathbf{U}^k \mathbf{\Sigma}^{k+1} (\mathbf{V}^k)^H, \tag{30}$$

where $\mathbf{U}^k \mathbf{\Lambda}^k (\mathbf{V}^k)^H$ is the full SVD of $\mathbf{W}^k$, $\mathbf{U}^k$ and $\mathbf{V}^k$ are unitary matrices, $\mathbf{\Lambda}^k = diag(\boldsymbol{\lambda}^k)$, $\boldsymbol{\lambda}^k = (\lambda_1^k, \lambda_2^k, \lambda_3^k)^T$ is the vector of singular values of $\mathbf{W}^k$ and satisfies $\lambda_1^k \ge \lambda_2^k \ge \lambda_3^k > 0$, and $\mathbf{\Sigma}^{k+1} = diag(\boldsymbol{\sigma}^{k+1})$, in which $\boldsymbol{\sigma}^{k+1} = (\sigma_1^{k+1}, \sigma_2^{k+1}, \sigma_3^{k+1})^T$ is defined as:

$$\boldsymbol{\sigma}^{k+1} = \max(1/\nu, \boldsymbol{\lambda}^k). \tag{31}$$

Proof. Let $\delta_j(\cdot)$ denotes the j-th largest singular value of a matrix, then we have:

$$\begin{aligned}&\mathcal{I}_\Phi(\mathbf{M}) + \frac{1}{2}\|\mathbf{M} - \mathbf{W}^k\|_F^2 \\ &= \mathcal{I}_\Phi(\mathbf{M}) + \frac{1}{2}(\|\mathbf{M}\|_F^2 - 2\langle \mathbf{M}, \mathbf{W}^k\rangle + \|\mathbf{W}^k\|_F^2) \\ &= \mathcal{I}_{\delta_3(\mathbf{M}) \ge 1/\nu} + \frac{1}{2}\left(\sum_{j=1}^3 \delta_j^2(\mathbf{M}) - 2\langle \mathbf{M}, \mathbf{W}^k\rangle + \sum_{j=1}^3 (\lambda_j^k)^2\right) \\ &\ge \mathcal{I}_{\delta_3(\mathbf{M}) \ge 1/\nu} + \frac{1}{2}\sum_{j=1}^3 \left(\delta_j^2(\mathbf{M}) - 2\delta_j(\mathbf{M})\lambda_j^k + (\lambda_j^k)^2\right) \\ &= \mathcal{I}_{\delta_3(\mathbf{M}) \ge 1/\nu} + \frac{1}{2}\sum_{j=1}^3 \left(\delta_j(\mathbf{M}) - \lambda_j^k\right)^2.\end{aligned} \tag{32}$$

The inequality in the fourth row of (32) is due to von Neumann's inequality [**?**], where the equality holds only when the matrices of left and right singular vectors of $\mathbf{M}$ are the same as those of $\mathbf{W}^k$. Then by minimizing the last row of (32), it is easy to see that the singular values of $\mathbf{M}$ are given by (31). Thus the theorem is proved. □

2 Solving Problem (8)

The solution process of problem (8) is similar to that of problem (7), which has been described in detail in the Appendix of the main body of the paper. To reduce redundancy, we adopt the notations used in solving problem (7) and only describe their differences in solution process.

Similarly, we reformulate problem (8) as follows:

$$\begin{aligned}&\min_{\mathbf{M}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{S}} \|\mathbf{M}^{-1}\|_2 + \mathcal{I}_\Theta(\mathbf{S}) \\ &\text{s.t. } \mathbf{M} = \mathbf{N}_1 + i\mathbf{N}_2, (\mathbf{N}_1 + i\mathbf{N}_2)\mathbf{a} = \mathbf{b}, \\ &\quad (\mathbf{C}_1\mathbf{N}_1 + \mathbf{D}_1\mathbf{N}_2)^T - \mathbf{1}/3 = \mathbf{S}, \mathbf{C}_2\mathbf{N}_1 + \mathbf{D}_2\mathbf{N}_2 = \mathbf{0},\end{aligned} \tag{33}$$

where $\Theta = \{\mathbf{S} | \|\mathbf{S}\|_{2,0} \leq \lfloor (1-\rho) n_r n_c \rfloor$ and $\mathbf{S} \geq -\mathbf{1}/3\}$.

The augmented Lagrangian function of problem (33) is:

$$\begin{aligned}
&\mathcal{L}(\mathbf{M}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{S}, \mathbf{X}, \mathbf{x}, \mathbf{Y}, \mathbf{Z}) \\
&= \|\mathbf{M}^{-1}\|_2 + \mathcal{I}_\Theta(\mathbf{S}) \\
&+ \langle \mathbf{X}, \mathbf{M} - (\mathbf{N}_1 + i\mathbf{N}_2) \rangle + \langle \mathbf{x}, (\mathbf{N}_1 + i\mathbf{N}_2)\mathbf{a} - \mathbf{b} \rangle \\
&+ \langle \mathbf{Y}, (\mathbf{C}_1\mathbf{N}_1 + \mathbf{D}_1\mathbf{N}_2)^T - \mathbf{1}/3 - \mathbf{S} \rangle + \langle \mathbf{Z}, \mathbf{C}_2\mathbf{N}_1 + \mathbf{D}_2\mathbf{N}_2 \rangle \\
&+ \frac{\beta}{2} \left(\|\mathbf{M} - (\mathbf{N}_1 + i\mathbf{N}_2)\|_F^2 + \|(\mathbf{N}_1 + i\mathbf{N}_2)\mathbf{a} - \mathbf{b}\|_2^2 \right. \\
&\quad \left. + \|(\mathbf{C}_1\mathbf{N}_1 + \mathbf{D}_1\mathbf{N}_2)^T - \mathbf{1}/3 - \mathbf{S}\|_F^2 + \|\mathbf{C}_2\mathbf{N}_1 + \mathbf{D}_2\mathbf{N}_2\|_F^2 \right).
\end{aligned} \tag{34}$$

Then by ADM, only the updates of $\mathbf{M}$ and $\mathbf{S}$ are different from those in solving problem (7). Here the two iterations are:

$$\begin{aligned}
\mathbf{M}^{k+1} &= \operatorname*{argmin}_{\mathbf{M}} \mathcal{L}(\mathbf{M}, \mathbf{N}_1^k, \mathbf{N}_2^k, \mathbf{S}^k, \mathbf{X}^k, \mathbf{x}^k, \mathbf{Y}^k, \mathbf{Z}^k) \\
&= \operatorname*{argmin}_{\mathbf{M}} \frac{1}{\beta}\|\mathbf{M}^{-1}\|_2 + \frac{1}{2}\|\mathbf{M} - \mathbf{W}^k\|_F^2,
\end{aligned} \tag{35}$$

$$\begin{aligned}
\mathbf{S}^{k+1} &= \operatorname*{argmin}_{\mathbf{S}} \mathcal{L}(\mathbf{M}^{k+1}, \mathbf{N}_1^{k+1}, \mathbf{N}_2^{k+1}, \mathbf{S}, \mathbf{X}^k, \mathbf{x}^k, \mathbf{Y}^k, \mathbf{Z}^k) \\
&= \operatorname*{argmin}_{\mathbf{S}} \mathcal{I}_\Theta(\mathbf{S}) + \frac{\beta}{2}\|\mathbf{S} - \mathbf{P}^k\|_F^2 \\
&= \operatorname*{argmin}_{\mathbf{S}} \mathcal{I}_\Theta(\mathbf{S}) + \frac{1}{2}\|\mathbf{S} - \mathbf{P}^k\|_F^2.
\end{aligned} \tag{36}$$

The solution to problem (35) is given by Theorem 2, which has been proven by Bai et al. [**?**].

Theorem 2 ([**?**]). *The solution to problem* (35) *is:*

$$\mathbf{M}^{k+1} = \mathbf{U}^k \mathbf{\Sigma}^{k+1} (\mathbf{V}^k)^H, \tag{37}$$

where $\mathbf{U}^k \mathbf{\Lambda}^k (\mathbf{V}^k)^H$ *is the full SVD of* $\mathbf{W}^k$*,* $\mathbf{U}^k$ *and* $\mathbf{V}^k$ *are unitary matrices,* $\mathbf{\Lambda}^k = diag(\boldsymbol{\lambda}^k)$*,* $\boldsymbol{\lambda}^k = (\lambda_1^k, \lambda_2^k, \lambda_3^k)^T$ *is the vector of singular values of* $\mathbf{W}^k$ *and satisfies* $\lambda_1^k \geq \lambda_2^k \geq \lambda_3^k > 0$*, and* $\mathbf{\Sigma}^{k+1} = diag(\boldsymbol{\sigma}^{k+1})$*, in which* $\boldsymbol{\sigma}^{k+1} = (\sigma_1^{k+1}, \sigma_2^{k+1}, \sigma_3^{k+1})^T$ *is the solution to the following problem:*

$$\min_{\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0} \frac{1}{\beta \sigma_3} + \frac{1}{2} \sum_{j=1}^{3} (\sigma_j - \lambda_j^k)^2. \tag{38}$$

Problem (36) can be solved by Algorithm 1.

Algorithm 1 The algorithm for problem (36)

Input: $\mathbf{P}^k$, ρ, n_r, n_c.

1: Compute $\mathbf{S} = \max(-\mathbf{1}/3, \Re(\mathbf{P}^k))$.
2: Compute the vector $\mathbf{r} \in \mathbb{R}^{n_r n_c \times 1}$, where the j-th element is defined as $[\mathbf{r}]_j = \|[\Re(\mathbf{P}^k)]_{:,j}\|_2$.
3: Sort $\mathbf{r}$ in descending order and have the permutation vector $\mathbf{q}$, which describes the element rearrangement of $\mathbf{r}$.
4: Set $[\mathbf{S}]_{:,[\mathbf{q}]_j} = \mathbf{0}$, $j \in \{\lfloor (1-\rho) n_r n_c \rfloor + 1, \cdots, n_r n_c\}$.

Output: $\mathbf{S}$.
